ON THIRD ORDER ROTATABLE DESIGNS WITH SMALLER NUMBER OF LEVELS

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1. INTRODUCTION

In recent years considerable emphasis has been laid for obtaining designs through which response surfaces can be explored. The rotatable designs of Box and Hunter (1957) and the designs for mixture experiments introduced by Scheffe (1963) are some of the more significant works in this line. Though the property of equal variance of estimated responses at equidistant points may not prove very useful in many situations, these designs do in fact supply us with good fractionally replicated design through which response surfaces can be investigated with relatively less material and cost. It may not be easy to construct such a fractional design through the ordinary technique of construction of fractionally replicated designs. From this viewpoint the number of points of any rotatable design has to be considerably smaller if they are to be useful as fractionally replicated designs in addition to the property of rotatability. A point of disadvantage is that often the number of levels of the factors is too large to be attractive to the experimenters.

In the present paper we have described some methods of construction of third order rotatable designs both sequential and non-sequential with five levels of each of the factors. In some cases the designs have reasonably small number of points. It has also been pointed out how through a design in v factors with t levels designs with any number of factors less than v can be obtained with number of points smaller than those of the design in v factors.

2. SERIES OF THIRD ORDER ROTATABLE DESIGNS

Das and Narasimham (1962) evolved a method that uses doubly balanced incomplete block designs to construct third order rotatable

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designs. The method consists of replacing the frequency 1 by an unknown a in the incidence matrix ($b^* \times v$) of the doubly balanced incomplete block design ($v, k, r, b^*, \lambda, \mu$) and then associating ± 1 to the unknown in all possible ways separately for each block ; then some more sets of combinations of the types $(b, o, \ldots, o)$, $(c, c, o, \ldots, o)$ and $(d, d, \ldots, d)$ are added to the $2^k b^*$ points so as to form a third order rotatable design. Once a third order rotatable design is obtained for v factors with N points, a series of such designs with as many levels for $(v-x)$ factors can be obtained. This is done by omitting x columns from the design for v factors considering it to be a $(N \times v)$ matrix. The numbers of points for these resulting third order rotatable designs are always less than those of the original design in v factors as some of the points become central points by virtue of the omissions of columns. The values of ratios of different unknowns remain unchanged. These results follow immediately from the fact that when some columns are omitted, the remaining columns still satisfy all the requirements of a third order design, as none of these excepting

$$\frac{\lambda 4}{\lambda^2_2} > \frac{v}{v+2} \text{ and } \frac{\lambda 2\, \lambda 6}{\lambda^2_4} > \frac{v+2}{v+4}$$

is a function of v. It is evident that these requirements will still be satisfied when v is replaced by $(v-x)$ once they hold for v or rather, if they do not hold for a certain v, they may still hold for $(v-x)$. This fact allows to obtain rotatable designs from rotatable arrangement by omitting certain factors or columns from the arrangements.

When $x=k$ the number of points can always be reduced considerably ; we can omit first from the rotatable design in v factors, the design points obtained from a block of the original doubly B.I.B. design in v factors. Next all those k columns which have numbers equal to those varietal numbers which are in the omitted block are to be taken out.

The technique holds good for getting sequential third order rotatable designs as well.

Das and Narasimham (1962) have reported a useful design for 11 factors. The design consists of

1056 points obtained from a-(11, 5, 15, 33, 6, 2),

22 ,, ,, ,, $(b, 0, \ldots, 0)$,

and 22 ,, ,, ,, $(c, 0, \ldots, 0)$.

This design may be used to give rotatable designs for 10, 9 and 8 factors with 1096, 1092 and 1088 points by omitting 1, 2 and 3 columns respectively. The values of

$$\frac{b^2}{a^2} \text{ and } \frac{c^2}{a^2}$$

remain the same for all these factors.

Another design for 5 factors obtained through

$$a\text{-}(5, 3, 6, 10, 3, 1), (b, 0, \ldots, 0) \text{ and } (c, 0, \ldots, 0)$$

with 100 points [reported by Das and Narasimham (1962)] may be exploited to give 7 level designs for 4 and 3 factors with just 96 and 92 points respectively.

3. Third Order Rotatable Designs in 5 Levels

The method of obtaining third order rotatable design in 5 levels is, in general, to take at least two suitably chosen sets from the following sets of combinations :

(*i*) $(a, a, \ldots, a,)$, (*ii*) $(b, b, \ldots, b,)$,

(*iii*) $(a, 0, \ldots, 0,)$, (*iv*) $(b, 0, \ldots, 0,)$,

(*v*) $(b, b, 0, \ldots, 0,)$, (*vi*) $(a, a, 0, \ldots, 0,)$,

and (*vii*) the a-combinations obtainable from B.I.B. designs either doubly balanced or complementary designs, and repeat them p, q, r etc. times. Next, through the relations of the third order designs, we shall be getting equations in terms of p, q, r, etc. and the positive integral solutions of p, q, r, etc., will give us the number of repetitions, necessary for the respective sets.

When $\lambda = 3\mu$ in a doubly B.I.B. design, with parameters $(v, k, r, b^*, \lambda, \mu)$ third order designs with five levels, can, in general, be obtained by taking the sets $(a, 0, \ldots, 0)$ and $(b, 0, \ldots, 0)$, repeated q and r times respectively, together with the a-combinations of the doubly B.I.B. design which are to be repeated p times.

For $\lambda \neq 3\mu$, no general method for the choice of such sets seems possible.

When a doubly B.I.B. design is not available, we take a B.I.B. design along with its complementary design so that the combined design becomes a doubly B.I.B. design.

Example. A non-sequential third order rotatable design in 4 factors can be obtained with the following points :

(*i*) $4 \times 6p$ points of a—$(4, 2, 3, 6, 1, 0) \times 2^2$

(*ii*) $8q$ points of $(a\ 0\ 0\ 0)\times 2$

(*iii*) $8r$ points of $(b\ 0\ 0\ 0)\times 2$

and (*iv*) $16m$ points of $(a\ a\ a\ a)\times 2^4$.

The three relations of the design give the following equations :

$$16ma^4+2rb^4+12pa^4+2qa^4=3[16ma^4+4pa^4],$$

$$16ma^6+2rb^6+12pa^6+2qa^6=5[16ma^6+4pa^6]$$

and

$$16ma^6+4pa^6=3[16ma^6].$$

The third equation gives

$$4p=16\times 2m$$

which is satisfied when $m=1$ and $p=8$. Putting these values in the first two equations and letting

$$\frac{b^2}{a^2}=S,$$

we get $$S^2=\frac{16-q}{r},\ S^3=\frac{64-q}{r}.$$

These equations are satisfied if $q=0$ and $r=1$, giving $S^2=16$ and $S^3=64$ which implies that

$$S=\frac{b^2}{a^2}=4.$$

Thus, we get a five level third order rotatable design in 216 points.

From the five level non-sequential third order rotatable designs, obtained as described earlier, we could obtain sequential designs by dividing the sets into two blocks such that each block is a second order rotatable design.

4. Series of Third Order Designs in Five Levels

After 5 level third order rotatable design has been constructed for v factors as described in the foregoing section, a series of such five level designs can be obtained for $(v-x)$ factors $0\leqslant x\geqslant v-3$ by just omitting x columns. The number of design points for these resulting designs for $(v-x)$ factors are always less than those of the design in v factors. These design points can further be reduced to a large extent if it is possible to divide the design points of a third order rotatable design in $(v-x)$ factors into two or more identical subgroups of design points. In such situations all these subgroups of points will constitute a third order rotatable design in five levels and any one of them can be taken. This is so because whatever relations are satisfied by two ore more identical groups of points collectively are also satisfied by each group individually.

The fact that rotatable arrangements can also be exploited to give rotatable designs for smaller number of factors as explained in section 2 also holds for 5 level designs. This may be illustrated by considering a third order rotatable arrangement for 8 factors obtained with the help of the *a*-combination of the doubly balanced incomplete block design (8, 4, 7, 14, 3, 1) and the set $(b, 0, \ldots, 0)$. This arrangement has 240 points in all. By omitting i columns $(i=1, 2, 3, 4, 5)$ from this we get third order rotatable designs for 7, 6, 5, 4 and 3 factors respectively with 238, 236, 234, 216 and 214 points respectively. The ratio $\frac{b^2}{a^2}$ of the unknown remains the same for all these factors.

Summary

A method of construction of third order rotatable designs both sequential and non-sequential in $(v-x)$ factors from a rotatable design in v factors by omitting x columns from the v factor design matrix $(N \times v)$ has been described. Through this technique rotatable arrangements can also be exploited to obtain rotatable designs for lesser number of factors. A method for constructing third order rotatable designs both sequential and non-sequential in five levels has also been presented. This technique of obtaining rotatable designs is useful particularly for getting designs with smaller number of points.

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REFERENCES

1. *Box, G.E.P. (1954).* The exploration and exploitation of response : surfaces Some general considerations and examples. Biometrics 10, 16-60.

2. *Box, G.E.P. and Hunter, J.S. (1957).* Multifactorial experimental designs for exploring response surfaces. Ann. Math. Stat. 28, 195-241.

3. *Calvin, L.D. (1954).* Doubly balanced incomplete block designs for experiments in which the treatment effects are correlated. Biometrics 10, 61-88.

4. *Das, M.N. (1961).* Construction of rotatable designs from factorial designs. J. Indian Soc. Agri. Stat. 13, 169-194.

5. *Das, M.N. and Narasimham, V.L. (1962).* Construction of rotatable designs through B.I.B. designs. Ann. Math. Stat. 33, 1421-1439.

6. *Das, M.N. (1963).* On construction of second order rotatable designs through B.I.B. designs with blocks of unequal sizes. Cal. Stat. Assocn. Bull. 12, 31-46.

7. *Draper, N.R. (1960).* Third order rotatable designs in three dimensions. Ann. Math. Stat, 31. 865-874.

8. *Draper, N.R. (1960).* A third order rotatable design in four dimensions. Ann. Maths, Stat. 31, 875-877.

9. *Gardiner, D.A., Grandage, A.H.E. and Hader, R.J. (1959).* Third order rotatable designs for exploring response surfaces. Ann. Math. Stat. 30, 1082-1096.

10. *Nigam, A.K. and Das, M.N. (1966).* On a method of construction of rotatable designs with smaller number of points controlling the number of levels. Calcutta Statistical Association Bulletin, Vol. 15, No. 60, 147-157.

11. *Scheffe, H. (1963).* The simplex-centroid design for experiments with mixtures.